Large Scale Systems

Chapter 6

Structuring

Decomposition

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Introduction

Large scale systems

- Model order reduction
- Control Structuring
  - System structuring
  - Sub-systems decomposition

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Large scale systems

- Model order reduction
- Control Structuring
  - System structuring
  - Sub-systems decomposition

Decomposition

- Natural, Application related
- Algorithms

Objective

Let consider the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

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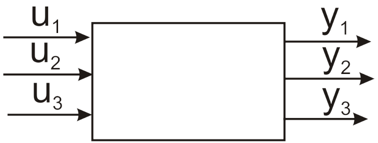
Input-Output decomposition

Set of single controllers

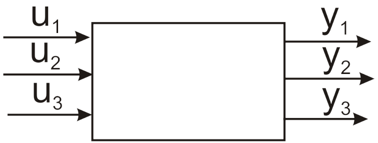
Input-Output Decomposition



SISO Controller

pairing: $u_i \to y_j$

Input-Output Decomposition



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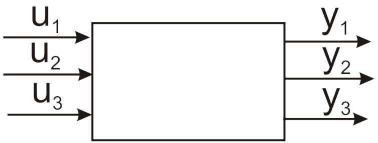
Assumptions

- Asymptotically stable system
- The same number of input and outputs

Input-Output Decomposition



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1. Relative Gain Array
2. Gramians & Participation Matrix

Relative Gain Array

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$$G(s) = C(sI - A)^{-1}B.$$

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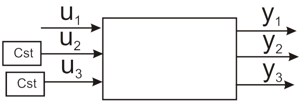
Interpretation

$$\lambda_{ij} = \frac{(\partial y_i/\partial u_j)_{u_{l\neq j} constant}}{(\partial y_i/\partial u_j)_{y_{k\neq i} constant}}$$

RGA Interpretation

$$\left(\partial y_i / \partial u_j\right)_{u_{l \neq j} constant}$$

variations of y_i with respect to u_i when other inputs u_l are constant.



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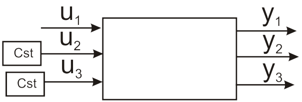
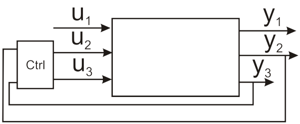
variations of y_i with respect to u_i when other outputs y_k are kept constants by other inputs u_l



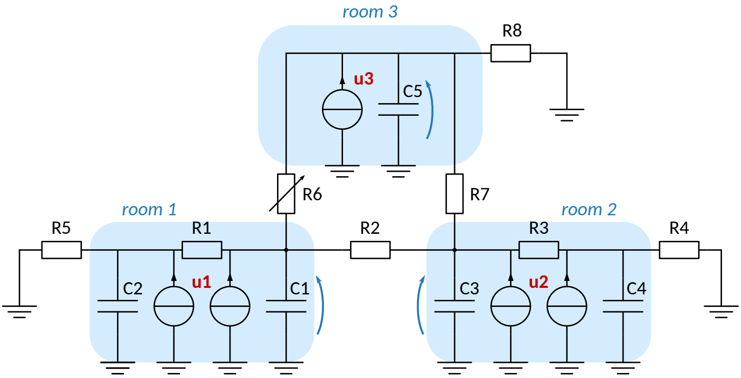


RGA: decomposition

Choice of pairs y_i, u_j:

- λ_{ij} is close to 1
- avoid negative values for λ_{ij}.

Example: Thermal model of a Building



3 cases: different values for R_6 (“opened vs. closed door”)

To DO - 1: Relative Gain Array

1. Consider the 3 systems ’Build’ saved as ’Building1.mat’, ’Building2.mat’ and ’Building3.mat’
2. Compute the relative gain arrays for these 3 systems
3. Conclude

Niederlinski Index

Let $\bar{G}_0$ the matrix deduced from G_0 keeping the entries for selected pairs y_i/u_j.

Definition: Niederlinski Index

$$N_g = \frac{det(G_0)}{det(\bar{G}_0)}$$

This index should be close to 1.

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To DO - 1: Relative Gain Array

4. Compute Niederlinsky index for the 3 decompositions
5. Conclude

Gramians & Participation Matrix

Controllability Gramian

Solution of

$$AW_c + W_cA^T + BB^T = 0$$

that characterizes input to state relationship

Gramians & Participation Matrix

Controllability Gramian

Solution of

$$AW_c + W_c A^T + BB^T = 0$$

that characterizes input to state relationship

Observability Gramian

Solution of

$$AW_0 + W_0 A^T + C^T C = 0$$

that characterizes state to output relationship let:

- b_j be the j^{th} column of B,
- c_i be the i^{eth} row of C,

Then let:

- $W_{c,j}$ the controllability gramian associated to A, b_j,
- $W_{o,i}$ the observability gramian associated to A, c_i.

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Properties

- $W_c = \sum_{j=1}^{m} W_{c,j}$
- $W_o = \sum_{i=1}^{m} W_{o,i}$
- $W_c W_o = \sum_{j=1}^{m} \sum_{i=1}^{m} W_{c,j} W_{o,i}$

Participation Matrix

Definition: Participation Matrix

Let the matrix P the entries of which are defined by:

$$P_{i,j} = \frac{trace\,(W_{c,j} W_{o,i})}{trace\,(W_c W_o)}$$

Property:

$$\sum_{j=1}^{m} \sum_{i=1}^{m} P_{i,j} = 1$$

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Decomposition

Select the pairs y_i/u_j corresponding to the greatest elements of P.

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Decomposition

Select the pairs y_i/u_j corresponding to the greatest elements of P.

To DO - 2: Participation Matrix

1. Compute the Participation matrix for the 3 systems
2. Conclude

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Let consider the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

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Objective

Find M sub-systems s.a.

$$\begin{aligned} \dot{x}_i &= A_{ii}x_i + B_{ii}u_i + \sum_{j\neq i}^{M} (A_{ij}x_j + B_{ij}u_j) \\ y_i &= C_{ii}x_i + \sum_{j\neq i}^{M} C_{ij}x_j \end{aligned}$$

where $\begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_M \end{pmatrix} = Perm(u)$ and $\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_M \end{pmatrix} = Perm(y)$

Structure representation

assumption

- A: $n - by - n$ matrix,
- B: $n - by - m$ matrix,
- C: $p - by - n$ matrix.

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Definition: Interconnection matrix

let

$$S = \begin{pmatrix} A & B & 0_{n\times p} \\ 0_{m\times n} & 0_{m\times m} & 0_{m\times p} \\ C & 0_{p\times m} & 0_{p\times p} \end{pmatrix}$$

The *Interconnection Matrix* $[S]$ is deduced by changing non-zero entries of S by 1.

Example:

Let the dynamical system:

$$\dot{x} = \begin{pmatrix} 3 & 0.1 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \\ 0.4 & -1 \end{pmatrix} x$$

Example:

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We get:

$$S = \left(\begin{array}{cc|cc|cc} 3 & 0.1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 5 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & -1 & 0 & 0 & 0 & 0 \end{array}\right) \quad [S] = \left(\begin{array}{cc|cc|cc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

Reachability

s-step reachability matrix

$$R_s = Structure([S] + [S]^2 + [S]^3 + \ldots + [S]^s)$$

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Reachability Matrix

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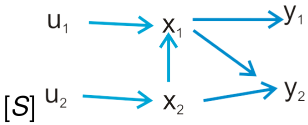
Example:

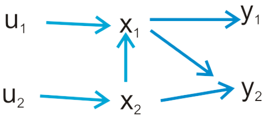
$$R_2 = \left(\begin{array}{cc|cc|cc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right) \qquad R = \left(\begin{array}{cc|cc|cc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right)$$

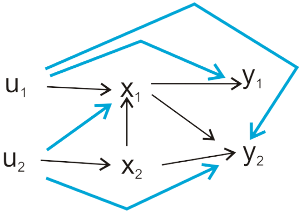
Interpretation



Interpretation

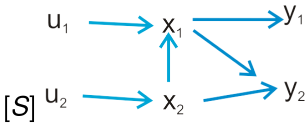
[S]

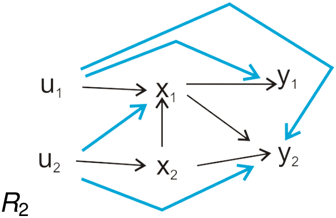


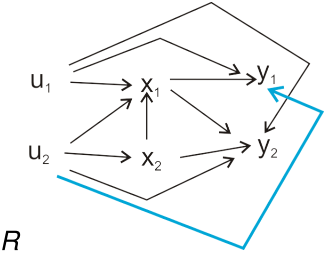


R_2

Interpretation







Reachability

Input reachablity

each state is influenced by at least one input.

Onput reachablity

each state influence at least one output.

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Interpretation

- Input reachability: controllability
- Output Reachability: Observability

Decomposition of R:

$$R = \begin{pmatrix} F & G & 0_{n\times p} \\ 0_{m\times n} & 0_{m\times m} & 0_{m\times p} \\ H & J & 0_{p\times p} \end{pmatrix}$$

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Conditions

- *Input Reachability* : G has no zero row,
- *Output Reachablity* : H has no zero column.

Example:

$$G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

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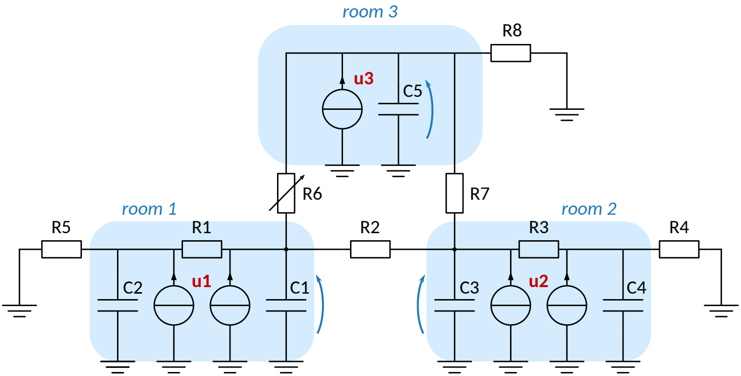
Example: Thermal model of a Building



To DO - 3: Reachability Matrix

1. Compute the Reachabilty matrix of the 3 systems
2. Comment

Objective

Objectif

Define M sub-systems

$$\dot{x}_i = A_{ii}x_i + B_{ii}u_i + \sum_{j\neq i}^{M} (A_{ij}x_j + B_{ij}u_j)$$

$$y_i = C_{ii}x_i + \sum_{j\neq i}^{M} C_{ij}x_j$$

where $\begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_M \end{pmatrix} = Perm(u)$ and $\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_M \end{pmatrix} = Perm(y)$

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\end{aligned}
$$

where $\begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_M \end{pmatrix} = Perm(u)$ and $\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_M \end{pmatrix} = Perm(y)$ and $\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_M \end{pmatrix} = Perm(x)$

Weakly coupled systems

We are searching for sub-systems where the interactions $\sum_{j\neq i}^{M}(A_{ij}x_j + B_{ij}u_j)$ et $\sum_{j\neq i}^{M} C_{ij}x_j$ are weak.

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Weak Interactions?

entries of matrices A_{ij}, B_{ij} et C_{ij} are small ($\leq \epsilon$)

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Procedure

- Build the matrix

$$S = \begin{pmatrix} A & B \\ C & 0_{p\times m} \end{pmatrix}$$

- change entries less (absolute value) than ϵ to 0
- determine the connex parts of the graph that define sub-systems
- compute state equations of the sub-systems

Example:

$$S = \left(\begin{array}{cc|cc} 3 & 0.1 & 1 & 0 \\ 0 & 2 & 0 & 5 \\ \hline 1 & 0 & 0 & 0 \\ 0.4 & -1 & 0 & 0 \end{array}\right) \qquad S_{\epsilon=0.5} = \left(\begin{array}{cc|cc} 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 5 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$S_{\epsilon=0.2} = \left(\begin{array}{cc|cc} 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 5 \\ \hline 1 & 0 & 0 & 0 \\ 0.4 & -1 & 0 & 0 \end{array}\right)$$

Verification

- sub-systems have inputs, states and outputs,
- sub-systems are *input reachable* and *output reachable*

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To DO - 4: ϵ-decomposition

1. Compute the decompositions for $\epsilon = 0.01$, $\epsilon = 0.05$ and $\epsilon = 0.2$ for the 3 buildings,
2. Conclude.